Quantization of the elastic modes in an isotropic plate

D. V. Anghel*and T. Kühn†

February 6, 2008

Abstract

We quantize the elastic modes in a plate. For this, we find a complete, orthogonal set of eigenfunctions of the elastic equations and we normalize them. These are the phonon modes in the plate and their specific forms and dispersion relations are manifested in low temperature experiments in ultra-thin membranes.

1 Introduction

Nowadays, the devices used in many high-sensitivity applications reached such a level of miniaturization, that the wavelength of the quantum quasiparticles used in their modeling is comparable to the dimensions of the device. The examples we are most familiar with are the ultra-sensitive electromagnetic radiation detectors. In a very general way of speaking, such detectors consist of some thin metallic films, a few tens of nanometers in thickness, deposited on a dielectric membrane. The dielectric membrane is usually made of SiN_x and has a thickness of the order of 100 nm. To reach the level of sensitivity and speed required for applications, these detectors have to work at sub-Kelvin temperatures and at such temperatures the dominant phonon wavelength is comparable to the devices' thickness [1, 2, 3, 4, 5, 6, 7, 8, 9].

To describe the thermal properties of such membranes or detectors, the electron-phonon interaction, or in general any interaction of phonons with impurities or disorder in the membrane, we have to know the phonon modes in the membrane. For this, we have to find the eigenmodes of the elastic equations and quantize the elastic field.

For infinite half-spaces, the quantization has been carried out by Ezawa [10].

*National Institute for Physics and Nuclear Engineering–"Horia Hulubei", Str. Atomistilor no.407, P.O.BOX MG-6, Bucharest - Magurele, Romania

†Nanoscience Center, Department of Physics, University of Jyväskylä, P.O. Box 35, FIN-40014 University of Jyväskylä

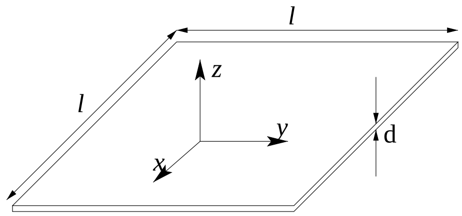

Figure 1: Typical plate, or membrane, like the ones used to support mesoscopic detectors. The thickness d is of the order of 100 nm and $l \gg d$. The plate surfaces are parallel to the xy plane and cut the z axis at $\pm d/2$.

1.1 Elastic eigenmodes in plates with parallel surfaces

The elastic eigenmodes in plates with parallel surfaces have been studied for a long time, mostly in connection with sound propagation and earthquakes (see for example Ref. [11]). To introduce these modes, let us consider a plate of thickness d and area l^2, with $l \gg d$. The two surfaces of the plate, or membrane, are parallel to the (xy) plane and cut the z axis at $\pm d/2$ (see Fig. 1). Throughout the paper we shall use V for the volume of the membrane (or in general of the solid that we describe–see Section 2.1) and ∂V for its surface. We shall assume that l is much bigger than any wavelength of the elastic perturbations considered here. The *displacement field* at *position* $\mathbf{r}$ is going to be denoted by $\mathbf{u}(\mathbf{r})$ or $\mathbf{v}(\mathbf{r})$. The unit vectors along the coordinate axes are denoted by $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

The displacement fields obey the dymanic equation

$$\rho \partial^2 u_i/\partial t^2 = c_{ijkl}\partial_j \partial_k u_l\,,\ \forall i = 1, 2, 3. \tag{1}$$

Here, as everywhere in this paper we shall assume summation over repeated indices. Assuming that the medium is isotropic (see for example Ref. [11] for the constraints on the tensor $[c]$), Eq. (1) is reduced to $\partial^2 u_i/\partial t^2 = \partial_j p_{ij}$, with p_{ij} defined as in [10], $p_{ij} = \rho^{-1} c_{ijkl}\partial_k u_l = (c_l^2 - 2c_t^2)(\partial_k u_k)\delta_{ij} + c_t^2(\partial_i u_j + \partial_j u_i)$, and c_l, c_t the longitudinal and transversal sound velocities, respectively.

Introducing the operator $\tilde{L}$ (we shall use *tilde* for operators and *hat* for unit vectors) by $\tilde{L}\mathbf{u} \equiv \rho^{-1} c_{ijkl}\partial_j\partial_k u_l = c_l^2 \mathrm{grad}\,\mathrm{div}\mathbf{u} - c_t^2 \mathrm{curl}\,\mathrm{curl}\mathbf{u} \equiv c_l^2 \nabla \cdot \nabla \cdot \mathbf{u} - c_t^2 \nabla \times \nabla \times \mathbf{u}$, the wave equation (1) becomes

$$\partial^2 \mathbf{u}/\partial t^2 = \tilde{L}\mathbf{u}. \tag{2}$$

The surface is free, so the stress should be zero there. This amounts to the boundary conditions [11, 10]

$$p_{ij} n_j = 0,\ \text{on } \partial V,\ \forall i = 1, 2, 3, \tag{3}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the surface, of components n_1, n_2, and n_3–*we shall use this notation throughout the paper.*

Applying Eqs. (2) and (3) to the plate we obtain the elastic eigenmodes, which are classified in three groups, according to their symmetry or polarization direction: the *horizontal shear* (h), the *symmetric* (s) and the *antisymmetric* (a) waves. All these waves are propagating (or decaying, if the wave-vector is complex) along the membrane and have a stationary form in the direction perpendicular to the surfaces. The h wave is polarized parallel to the surfaces and perpendicular to the propagation direction. The s and a waves are superpositions of longitudinal and transversal waves, polarized in a plane that is perpendicular to the surfaces and contains the propagation direction. The difference between the s and the a waves comes from the fact that the displacement field along the z direction is symmetric for the s wave and antisymmetric for the a wave, while the displacement field along the propagation direction is antisymmetric for the s wave and symmetric for the a wave (see below). Explicitely, the three types of modes are:

$$\begin{aligned}
\mathbf{u}_h &= (\hat{\mathbf{k}}_\parallel \times \hat{\mathbf{z}}) \cdot N_h \cos\left[\frac{m\pi}{b}\left(z - \frac{b}{2}\right)\right] e^{i(\mathbf{k}_\parallel\cdot\mathbf{r}-\omega t)}, \quad m = 0, 1, 2, \ldots, &\text{(4a)}\\
\mathbf{u}_s &= N_s \left\{ \hat{\mathbf{z}} \cdot k_\parallel \left[-2k_t k_l \cos\left(k_t \frac{b}{2}\right) \sin(k_l z) + [k_t^2 - k_\parallel^2] \cos\left(k_l \frac{b}{2}\right) \sin(k_t z) \right] \right.\\
&\quad \left. + \hat{\mathbf{k}}_\parallel \cdot i k_t \left[2k_\parallel^2 \cos\left(k_t \frac{b}{2}\right) \cos(k_l z) + [k_t^2 - k_\parallel^2] \cos\left(k_l \frac{b}{2}\right) \cos(k_t z) \right] \right\}\\
&\quad \times e^{i(\mathbf{k}_\parallel\cdot\mathbf{r}-\omega t)}, &\text{(4b)}\\
\mathbf{u}_a &= N_a \left\{ \hat{\mathbf{z}} \cdot k_\parallel \left[2k_t k_l \sin(k_t \frac{b}{2}) \cos(k_l z) - [k_t^2 - k_\parallel^2] \sin(k_l \frac{b}{2}) \cos(k_t z) \right] \right.\\
&\quad \left. + \hat{\mathbf{k}}_\parallel \cdot i k_t \left[2k_\parallel^2 \sin(k_t \frac{b}{2}) \sin(k_l z) + [k_t^2 - k_\parallel^2] \sin(k_l \frac{b}{2}) \sin(k_t z) \right] \right\}\\
&\quad \times e^{i(\mathbf{k}_\parallel\cdot\mathbf{r}-\omega t)}, &\text{(4c)}
\end{aligned}$$

where $\hat{\mathbf{k}}_\parallel$ is the unit vector along the propagation direction and $\mathbf{k}_\parallel = k_\parallel \cdot \hat{\mathbf{k}}_\parallel$. In the s and a modes, the wavevector in the z direction takes two values, k_t and k_l, one corresponding to the transversal component of the mode, the other one corresponding to the longitudinal component. The constants N_h, N_s, and N_a are the normalization constants, which will be calculated in Section 3.

The components of the wavevectors, k_t, k_l, and $k_\parallel$ obey the transcendental equations [11],

$$\frac{\tan(\frac{d}{2}k_t)}{\tan(\frac{d}{2}k_l)} = -\frac{4k_l k_t k_\parallel^2}{[k_t^2 - k_\parallel^2]^2} \tag{5a}$$

and

$$\frac{\tan(\frac{d}{2}k_t)}{\tan(\frac{d}{2}k_l)} = -\frac{[k_t^2 - k_\parallel^2]^2}{4k_l k_t k_\parallel^2}. \tag{5b}$$

for symmetric and antisymmetric waves, respectively.

All the solutions of the elastic equation (2) are given by equations (4a), (4b), and (4c), with m taking all the natural values, $0, 1, \ldots$, whereas k_t and k_l are the solutions of Eqs. (5a) and (5b). $k_\parallel$ can be a complex number, but the complete, orthogonal set of phonon modes that we shall use in the quantization of the elastic field are the ones with $k_\parallel$ running from 0 to ∞

The paper is organized as follows: in section 2.1 we show that $\tilde{L}$ is self-adjoint even when applied to the displacement field of an elastic body of arbitrary shape. Therefore, we can form a complete, orthonormal set of its eigenfunctions.

The fact that the elastic modes (4) corresponding to different quantum numbers are orthogonal to each other is proved in Section 2.2, based on the hermiticity of $\tilde{L}$.

The normalization constants are calculated in Section 3 and the formal procedure of quantising the elastic field is presented in Section 4.

We apply this formalism elsewhere to calculate the phonon scattering in amorphous, thin membranes.

2 Orthogonality and completeness of the set of elastic eigenmodes

2.1 Self-adjointness of the operator $\tilde{L}$

We shall prove the self-adjointness of $\tilde{L}$ on an arbitrary volume V. We assume that V has the smooth border, ∂V. The operator $\tilde{L}$ acts on the Hilbert space $\mathcal{H}$ which consists of the vector functions defined on V, which are integrable in modulus square. The scalar product on $\mathcal{H}$ is defined as usual,

$$\langle \mathbf{v}|\mathbf{u}\rangle = \int_V \mathbf{v}^\dagger(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r})\, d^3\mathbf{r}\,, \tag{6}$$

and the norm is $||\mathbf{u}|| \equiv \langle \mathbf{u}|\mathbf{u}\rangle^{1/2}$. The domain of $\tilde{L}$, denoted by $\mathcal{D}(\tilde{L})$, is formed of such functions $\mathbf{u}(\mathbf{r}) \in \mathcal{H}$, so that $\tilde{L}\mathbf{u}(\mathbf{r})$ exists and is contained in $\mathcal{H}$. Moreover, the functions from $\mathcal{D}(\tilde{L})$ should obey the boundary conditions (3). Ezawa showed that $\tilde{L}$ is hermitian [10]. We will show now that L is also self-adjoint.

First of all, for a formal treatment, in this section we will understand the derivatives in a generalized sense. If $f(\mathbf{r})$ is an arbitrary function on V, then $\partial_i f$ is defined as the function that satisfies

$$\int_V g(\mathbf{r})\partial_i f(\mathbf{r})\, d^3\mathbf{r} = \int_{\partial V} f(\mathbf{r})g(\mathbf{r})n_i\, d^2\mathbf{r} - \int_V f(\mathbf{r})\partial_i g(\mathbf{r})\, d^3\mathbf{r}\,, \tag{7}$$

for *any* function $g(\mathbf{r})$ of class $C^1(V)$ (i.e. $g(\mathbf{r})$ is derivable, with continous first derivatives on V) and all the integrals on the right-hand side of Eq. (7) exist and are finite.

Returning to our operator, let us first note that the space $\mathcal{D}(\tilde{L})$ includes the space of functions twice derivable, with continous, integrable, second derivatives, $C^2(V)$, which is dense in $\mathcal{H}$. Therefore $\mathcal{D}(\tilde{L})$ is also dense in $\mathcal{H}$, and $\tilde{L}$ is a

symmetric operator. Then we define the adjoint operator $\tilde{L}^\dagger$ and its domain $\mathcal{D}(\tilde{L}^\dagger)$. For this, let $\mathbf{v}$ be a function in $\mathcal{H}$, so that $\langle \mathbf{v}|\tilde{L}\mathbf{u}\rangle$, is a continous linear functional in $\mathbf{u} \in \mathcal{D}(\tilde{L})$, i.e. there exists an $M_\mathbf{v} > 0$ for which

$$\langle \mathbf{v}|\tilde{L}\mathbf{u}\rangle \leq M_\mathbf{v}||\mathbf{u}|| \tag{8}$$

for any $\mathbf{u} \in \mathcal{D}(\tilde{L})$. By the Riesz-Fréchet theorem [12], there exists a $\mathbf{v}^* \in \mathcal{H}$ so that $\langle \mathbf{v}|\tilde{L}\mathbf{u}\rangle = \langle \mathbf{v}^*|\mathbf{u}\rangle$ for any $\mathbf{u} \in \mathcal{D}(\tilde{L})$. The adjoint operator $\tilde{L}^\dagger$ is defined by the relation $\tilde{L}^\dagger\mathbf{v} = \mathbf{v}^*$, for all $\mathbf{v}$ that satisfy (8). The functions that satisfy (8) form the domain of $\tilde{L}^\dagger$, denoted $\mathcal{D}(\tilde{L}^\dagger)$. Since $\tilde{L}$ is hermitian, if $\mathbf{v}, \mathbf{u} \in \mathcal{D}(\tilde{L})$, then $\langle \mathbf{v}|\tilde{L}\mathbf{u}\rangle = \langle \tilde{L}\mathbf{v}|\mathbf{u}\rangle$ *is* a linear functional, so any function from $\mathcal{D}(\tilde{L})$ is included in $\mathcal{D}(\tilde{L}^\dagger)$. Therefore we can write $\mathcal{D}(\tilde{L}) \subset \mathcal{D}(\tilde{L}^\dagger)$. To prove that $\tilde{L}$ is self-adjoint, we have to show that also $\mathcal{D}(\tilde{L}^\dagger) \subset \mathcal{D}(\tilde{L})$, so in the end $\mathcal{D}(\tilde{L}^\dagger) = \mathcal{D}(\tilde{L})$ and $\tilde{L} = \tilde{L}^\dagger$.

For this, let us take $\mathbf{v} \in \mathcal{D}(\tilde{L}^\dagger)$ and $\mathbf{u} \in \mathcal{D}(\tilde{L})$. Integrating by parts we get

$$\begin{aligned}
\rho\langle \mathbf{v}|\tilde{L}\mathbf{u}\rangle &= \int_V v_i^\star c_{ijkl}\partial_j\partial_k u_l d^3\mathbf{r} &(9)\\
&= -\int_{\partial V}(\partial_j v_i^\star)c_{ijkl}n_k u_l d^2\mathbf{r} + \int_V(\partial_k\partial_j v_i^\star)c_{ijkl}u_l d^3\mathbf{r} \\
&= -\int_{\partial V}(\partial_j v_i^\star)c_{ijkl}n_k u_l d^2\mathbf{r} + \rho\langle \tilde{L}\mathbf{v}|\mathbf{u}\rangle \equiv \rho\langle \tilde{L}^\dagger\mathbf{v}|\mathbf{u}\rangle\,, &(10)
\end{aligned}$$

where we used $c_{ijkl}n_j\partial_k u_l = 0$ on ∂V (Eq. 3), and the simplified notation

$$\rho\langle \tilde{L}\mathbf{v}|\mathbf{u}\rangle \equiv \int_V (c_{ijkl}\partial_j\partial_k v_i^\star)u_l d^3\mathbf{r} = \int_V (c_{lkji}\partial_k\partial_j v_i^\star)u_l d^3\mathbf{r}\,, \tag{11}$$

although $\mathbf{v}$ is not necesarily a function in $\mathcal{D}(\tilde{L})$. The last equality in Eq. (11) is obtained using $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$ [11] and permuting the partial derivatives. But Eq. (10) means that

$$\langle(\tilde{L} - \tilde{L}^\dagger)\mathbf{v}|\mathbf{u}\rangle = \rho^{-1}\int_{\partial V}\partial_j v_i^\star n_k c_{ijkl}u_l d^2\mathbf{r} \tag{12}$$

for any $\mathbf{u} \in \mathcal{D}(\tilde{L})$. If $(\partial_j v_i^\star)n_k c_{ijkl}$ is not identically zero on ∂V, then we can find $\mathbf{u}$ so that the surface integral in (12) is different from zero. This implies that $\langle(\tilde{L} - \tilde{L}^\dagger)\mathbf{v}|\mathbf{u}\rangle \neq 0$, so $\mathbf{v} \neq 0$ on a set of measure larger than zero in the interior of V, denoted as V°. In such a case, we can find a nonempty compact set $S \subset V^\circ$ and a function, $\mathbf{u}'$, which is twice derivable, zero outside S, and satisfies

$$\langle(\tilde{L} - \tilde{L}^\dagger)\mathbf{v}|\mathbf{u}'\rangle \neq 0. \tag{13}$$

Since $\mathbf{u}'$ is zero outside S and S is a compact set in V°, this means that both, $\mathbf{u}'$ and any of its derivatives are zero on ∂V; therefore $\mathbf{u}' \in \mathcal{D}(\tilde{L})$. Now, $\mathbf{u}'(\mathbf{r} \in \partial V) = 0$, implies that

$$\int_{\partial V}\partial_j v_i^\star n_k c_{ijkl}(u_l + u_l')d^2\mathbf{r} = \int_{\partial V}\partial_j v_i^\star n_k c_{ijkl}u_l d^2\mathbf{r} \tag{14}$$

whereas (13) implies that

$$\langle(\tilde{L}-\tilde{L}^\dagger)\mathbf{v}|(\mathbf{u}+\mathbf{u}')\rangle \neq \langle(\tilde{L}-\tilde{L}^\dagger)\mathbf{v}|\mathbf{u}\rangle. \tag{15}$$

Since Eq. (12) should be valid for any function in $\mathcal{D}(\tilde{L})$, including $\mathbf{u}$ and $\mathbf{u}+\mathbf{u}'$ and Eq. (14) is true by construction, this implies that (15) is a contradiction; therefore $(\partial_j v_i^\star) n_k c_{ijkl} = 0$ on ∂V. Moreover, by the definition of $\mathcal{D}(\tilde{L}^\dagger)$ and the Riesz-Fréchet theorem, $c_{lkji}\partial_k\partial_j v_i \equiv \rho\tilde{L}^\dagger \mathbf{v} \in \mathcal{H}$. Therefore $\mathbf{v} \in \mathcal{D}(\tilde{L})$, so $\mathcal{D}(\tilde{L}^\dagger) = \mathcal{D}(\tilde{L})$ and $\tilde{L} = \tilde{L}^\dagger$, i.e. the operator $\tilde{L}$ is self-adjoint on an arbitrary volume, V.

Going back to the plate with infinite lateral extension, we have to find from the wave functions of the form (4) a complete, orthonormal set. We can do that by using the operator $\tilde{\mathbf{k}}_\parallel \equiv i(\partial_x + \partial_y)$, which is also self-adjoint when acting on wave-functions defined on a plate with infinite lateral extension, or on a finite, rectangular plate, with periodic boundary conditions at the edges. Since $\tilde{L}$ and $\tilde{\mathbf{k}}_\parallel$ commute and they are both self-adjoint operators, if we find *all* the eigenfunctions common to $\tilde{L}$ and $\tilde{\mathbf{k}}_\parallel$, then we have a complete set. But these functions are given by Eqs. (4), with real $k_\parallel$, and k_t and k_l satisfying (5).

2.2 Orthogonality of the elastic eigenmodes

Now we study the orthogonality of the elastic eigenmodes of the plate. For this, we write the the functions that appear in Eqs. (4a)-(4c) in general as $\mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r}) \equiv \mathbf{u}_{k_t,\sigma}(z)e^{i\mathbf{k}_\parallel\cdot\mathbf{r}}$, where we separated the x and y dependence of the fields from the z dependence and we disregarded the time dependence. By σ we denote the "polarization" h, s, or a. We shall not use k_l explicitely in the notations below, since it is determined implicitely by k_t, $k_\parallel$, and Eqs. (5). The operator $\tilde{\mathbf{k}}_\parallel$ has eigenvectors of the form $\mathbf{g}(z)e^{i\mathbf{k}_\parallel\cdot\mathbf{r}}$, where $\mathbf{g}(z)$ is an arbitrary function of z while the eigenvalue $\mathbf{k}_\parallel$, perpendicular on z, has real components along the x and y directions. We analyse the common set of eigenvectors of $\tilde{L}$ and $\tilde{\mathbf{k}}_\parallel$, so in what follows we shall consider only the cases with real wave-vectors $\mathbf{k}_\parallel$. First we observe that

$$\langle\mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma}|\mathbf{u}_{\mathbf{k}'_\parallel,k'_t,\sigma'}\rangle = 2\pi\delta(\mathbf{k}_\parallel-\mathbf{k}'_\parallel)\int_{-b/2}^{b/2}\mathbf{u}^\dagger_{k_t,\sigma}(z)\mathbf{u}_{k'_t,\sigma'}(z)\,dz, \tag{16}$$

so we are left to study the orthogonality of functions with the same $\mathbf{k}_\parallel$. For simplicity we choose $\mathbf{k}_\parallel = \hat{\mathbf{x}}\cdot k_\parallel$, so the h waves are polarized in the $\hat{\mathbf{y}}$ direction and the s and a waves have displacement fields in the (xz) plane. Since the displacement field of the h waves are perpendicular to the displacement fields of the s and a waves, any h wave is orthogonal to any s or a wave. Similarly, for the same $\mathbf{k}_\parallel$, the displacement fields of any of the s and a waves, although in the same plane, are orthogonal to each-other due to their opposite symmetries. We conclude that

$$\int_{-b/2}^{b/2}\mathbf{u}^\dagger_{k_t,\sigma}(z)\mathbf{u}_{k'_t,\sigma'}(z)\,dz = 0\,,$$

for any k_t and k'_t, if $\sigma \neq \sigma'$. Therefore $\langle \mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma} | \mathbf{u}_{\mathbf{k}'_\parallel,k'_t,\sigma'} \rangle \propto \delta(\mathbf{k}_\parallel - \mathbf{k}'_\parallel)\delta_{\sigma,\sigma'}$.

We are left to show that $\langle \mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma} | \mathbf{u}_{\mathbf{k}_\parallel,k'_t,\sigma} \rangle \propto \delta_{k_t,k'_t}$ which follows simply from the hermiticity of the operator $\tilde{L}$. We calculate the matrix element

$$\begin{aligned}\langle \mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma} | \tilde{L} \mathbf{u}_{\mathbf{k}_\parallel,k'_t,\sigma} \rangle &= -\omega^2_{\mathbf{k}_\parallel,k'_t,\sigma} \langle \mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma} | \mathbf{u}_{\mathbf{k}_\parallel,k'_t,\sigma} \rangle \\ &= \langle \tilde{L} \mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma} | \mathbf{u}_{\mathbf{k}_\parallel,k'_t,\sigma} \rangle = -\omega^2_{\mathbf{k}_\parallel,k_t,\sigma} \langle \mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma} | \mathbf{u}_{\mathbf{k}_\parallel,k'_t,\sigma} \rangle\end{aligned}$$

But since for given $\mathbf{k}_\parallel$ and σ, the eigenstates of $\tilde{L}$ are not degenerate, $\langle \mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma} | \mathbf{u}_{\mathbf{k}_\parallel,k'_t,\sigma} \rangle = 0$, unless $k_t = k'_t$ and, of course, $k_l = k'_l$. This completes the proof and

$$\langle \mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma} | \mathbf{u}_{\mathbf{k}'_\parallel,k'_t,\sigma'} \rangle \propto \delta_{\sigma,\sigma'} \delta(\mathbf{k}_\parallel - \mathbf{k}'_\parallel) \delta_{k_t,k'_t} \,. \tag{17}$$

In the next section we calculate the normalization constants.

3 Normalization of the elastic modes

From this point on we shall assume that the plate has finite lateral extensions of area $A = l \times l$ and the eigenfunctions obey periodic boundary conditions in the (xy) plane. The volume of the plate is $V = Ad$. As a consequence, the values of $\mathbf{k}_\parallel$ are discrete and the scalar product (17) should give

$$\langle \mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma} | \mathbf{u}_{\mathbf{k}'_\parallel,k'_t,\sigma'} \rangle = \delta_{\sigma,\sigma'} \delta_{\mathbf{k}_\parallel,\mathbf{k}'_\parallel} \delta_{k_t,k'_t} \,. \tag{18}$$

For horizontal shear waves, the normalization constant is simple to calculate. In this case, let us write $\mathbf{u}_{\mathbf{k}_\parallel,k_t,h}$ simply as $\mathbf{u}_{\mathbf{k}_\parallel,m,h}$ (see 4a) and we obtain

$$\begin{aligned}||\mathbf{u}_{\mathbf{k}_\parallel,m,h}||^2 &= (N_{\mathbf{k}_\parallel,m,h})^2 \frac{V}{2}, \qquad \text{for } m > 0, &\text{(19a)} \\ &= (N_{\mathbf{k}_\parallel,m,h})^2 V, \qquad \text{for } m = 0. &\text{(19b)}\end{aligned}$$

So $N_{\mathbf{k}_\parallel,0,h} = V^{-1/2}$ and $N_{\mathbf{k}_\parallel,m>0,h} = (2/V)^{1/2}$.

For symmetric Lamb modes, from Eq. (5b) we calculate k_l and k_t as functions of $k_\parallel$. The results are shown in Fig. 2 (a). As expected, for each value of $k_\parallel$, the wave-vectors k_l and k_t take only discrete values, but not as simple as the values corresponding to the h modes (4a). Each curve in Fig. 2 (a) corresponds to a diffferent branch of the dispersion relation, $\omega_{k_\parallel,k_t(k_\parallel,m),s}$, where $m = 0, 1, \ldots$ denotes the branch number. Branches with bigger m are placed above branches with smaller m.

As $k_\parallel$ increases, $k_l(k_\parallel, m)$ decreases and, after reaching the value 0, turns imaginary. On the other hand, $k_t(k_\parallel, m)$ first increases with $k_\parallel$, reaches a maximal value and then decreases monotonically as $k_\parallel$ increases to infinity. Its decrease is bounded for all branches, except the lowest one, where, after reaching zero at some finite $k_\parallel$, it turns imaginary in the lower-left quadrant of Fig. 2 (a). For the clarity of the calculations, the imaginary values of k_l and k_t are denoted as $i\kappa_l$ and $i\kappa_t$, respectively, where κ_l and κ_t take real, positive values. Both, κ_l and κ_t increase without limit as $k_\parallel$ increases to infinity.

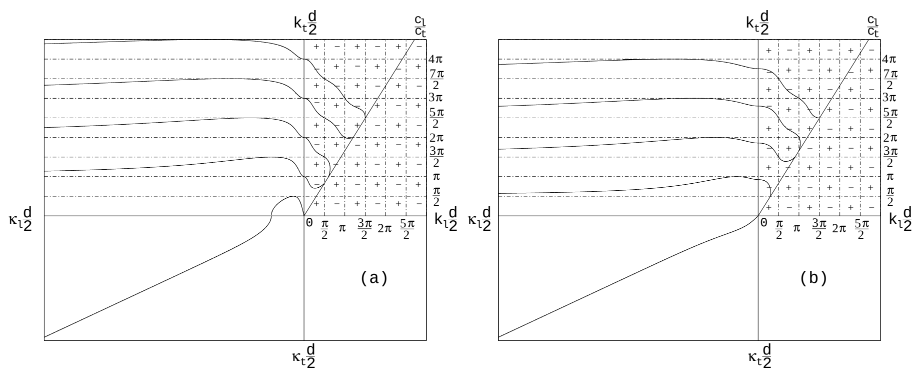

Figure 2: The curves $(k_l(k_\parallel), k_t(k_\parallel))$ for symmetric (a) and antisymmetric (b) Lamb modes. On each curve $k_\parallel$ varies from 0 to ∞, as we go from its right end to the left. The different curves correspond to different branches of the dispersion relation, i.e. to different values of k_l and k_t at $k_\parallel = 0$. Both, k_l and k_t may take real and imaginary values. The imaginary values of k_l, denoted by κ_l are to the right of the vertical axis and the imaginary values of k_t, denoted by κ_t are below the horizontal axis in both figures. As in the rest of the paper, d is the thickness of the plate. For these plots we used $c_l/c_t = 1.66$, corresponding to SiN_x plates. The + and − signs in the upper-right quadrants are only for orientation purposes.

We encounter a similar situation for the antisymmetric Lamb modes (see Fig. 2 b). The only marked differences between the symmetric and antisymmetric modes are the following: the asymptotic values of $(d/2)\cdot k_t(k_\parallel \to \infty, m)$ are $\pi, 2\pi, \ldots$ for symmetric modes and $\pi/2, 3\pi/2, \ldots$ for antisymmetric modes; the maxima of $(d/2)\cdot k_t(k_\parallel, m)$ are $(2m+1)\pi/2$ for the symmetric modes and $m\pi$ for the antisymmetric modes. Also, for the antisymmetric modes $k_t(k_\parallel, 0)$ (i.e. the lowest branch) takes only imaginary values.

Integrating $|\mathbf{u}_s|^2$ over the volume of the plate and using the transcendental equation (5a), together with Snell's law, $\omega^2 = c_t^2(k_t^2 + k_\parallel^2) = c_l^2(k_l^2 + k_\parallel^2)$, we obtain the normalization constant for the symmetric modes in the *quadrant I* (upper-right) of Fig. 2 (a)

$$\begin{aligned}(N_s^I)^{-2} \quad = \quad & A\Bigg\{4k_t^2k_\parallel^2\cos^2(k_td/2)\left((k_l^2+k_\parallel^2)\frac{d}{2} - (k_l^2-k_\parallel^2)\frac{\sin(k_ld)}{2k_l}\right) \\ & +(k_t^2-k_\parallel^2)^2\cos^2(k_ld/2)\left((k_t^2+k_\parallel^2)\frac{d}{2} + (k_t^2-k_\parallel^2)\frac{\sin(k_td)}{2k_t}\right) \\ & +4k_tk_\parallel^2(k_t^2-k_\parallel^2)\cos^2(k_ld/2)\sin(k_td)\Bigg\} \qquad (20a)\end{aligned}$$

in the *quadrant II* (upper-left),

$$
\begin{aligned}
(N_s^{II})^{-2} \quad &= \quad A\left\{4k_t^2k_\parallel^2\cos^2(k_td/2)\left((\kappa_l^2+k_\parallel^2)\frac{\sinh(\kappa_ld)}{2\kappa_l}-(\kappa_l^2-k_\parallel^2)\frac{d}{2}\right)\right. \\
&\quad +(k_t^2-k_\parallel^2)^2\cosh^2(\kappa_ld/2)\left((k_t^2+k_\parallel^2)\frac{d}{2}+(k_t^2-k_\parallel^2)\frac{\sin(k_td)}{2k_t}\right) \\
&\quad \left.+4k_tk_\parallel^2(k_t^2-k_\parallel^2)\cosh^2(\kappa_ld/2)\sin(k_td)\right\} \qquad (20b)
\end{aligned}
$$

and in the *quadrant III* (lower-left),

$$
\begin{aligned}
(N_s^{III})^{-2} \quad &= \quad A\left\{4\kappa_t^2k_\parallel^2\cosh^2(\kappa_td/2)\left((\kappa_l^2+k_\parallel^2)\frac{\sinh(\kappa_ld)}{2\kappa_l}-(\kappa_l^2-k_\parallel^2)\frac{d}{2}\right)\right. \\
&\quad +(\kappa_t^2+k_\parallel^2)^2\cosh^2(\kappa_ld/2)\left((\kappa_t^2+k_\parallel^2)\frac{\sinh(\kappa_td)}{2\kappa_t}+(\kappa_t^2-k_\parallel^2)\frac{d}{2}\right) \\
&\quad \left.-4\kappa_tk_\parallel^2(\kappa_t^2+k_\parallel^2)\cosh^2(\kappa_ld/2)\sinh(\kappa_td)\right\} \qquad (20c)
\end{aligned}
$$

Similarly, the normalization constants for symmetric modes in the three quadrants are

$$
\begin{aligned}
(N_a^{I})^{-2} \quad &= \quad A\left\{4k_t^2k_\parallel^2\sin^2(k_td/2)\left((k_l^2+k_\parallel^2)\frac{d}{2}+(k_l^2-k_\parallel^2)\frac{\sin(k_ld)}{2k_l}\right)\right. \\
&\quad +(k_t^2-k_\parallel^2)^2\sin^2(k_ld/2)\left((k_t^2+k_\parallel^2)\frac{d}{2}-(k_t^2-k_\parallel^2)\frac{\sin(k_td)}{2k_t}\right) \\
&\quad \left.-4k_tk_\parallel^2(k_t^2-k_\parallel^2)\sin^2(k_ld/2)\sin(k_td)\right\} \qquad (21a)
\end{aligned}
$$

$$
\begin{aligned}
(N_a^{II})^{-2} \quad &= \quad A\left\{4k_t^2k_\parallel^2\sin^2(k_td/2)\left((\kappa_l^2+k_\parallel^2)\frac{\sinh(\kappa_ld)}{2\kappa_l}+(\kappa_l^2-k_\parallel^2)\frac{d}{2}\right)\right. \\
&\quad +(k_t^2-k_\parallel^2)^2\sinh^2(\kappa_ld/2)\left((k_t^2+k_\parallel^2)\frac{d}{2}-(k_t^2-k_\parallel^2)\frac{\sin(k_td)}{2k_t}\right) \\
&\quad \left.-4k_tk_\parallel^2(k_t^2-k_\parallel^2)\sinh^2(\kappa_ld/2)\sin(k_td)\right\} \qquad (21b)
\end{aligned}
$$

and

$$
\begin{aligned}
(N_a^{III})^{-2} \quad &= \quad A\left\{4\kappa_t^2k_\parallel^2\sinh^2(\kappa_td/2)\left((\kappa_l^2+k_\parallel^2)\frac{\sinh(\kappa_ld)}{2\kappa_l}+(\kappa_l^2-k_\parallel^2)\frac{d}{2}\right)\right. \\
&\quad +(\kappa_t^2+k_\parallel^2)^2\sinh^2(\kappa_ld/2)\left((\kappa_t^2+k_\parallel^2)\frac{\sinh(\kappa_td)}{2\kappa_t}-(\kappa_t^2-k_\parallel^2)\frac{d}{2}\right) \\
&\quad \left.-4\kappa_tk_\parallel^2(\kappa_t^2+k_\parallel^2)\sinh^2(\kappa_ld/2)\sinh(\kappa_td)\right\} \qquad (21c)
\end{aligned}
$$

As one can notice, in the quadrants II and III the trigonometric functions are replaced by hyperbolic functions because of the change $k_l \to i\kappa_l$–in quadrant II–and $k_l \to i\kappa_l$, $k_t \to i\kappa_t$–in quadrant III.

Although Eqs. (20) and (21), with all their cases, are convenient because they are ready to use in practical calculations, we shall pack each set in one equation with the help of the complex wave-vector components $\bar{k}_t \equiv k_t + i\kappa_t$ and $\bar{k}_l \equiv k_l + i\kappa_l$:

$$\begin{aligned} N_s^{-2} &= A \left\{ 4|\bar{k}_t|^2 k_\parallel^2 |\cos(\bar{k}_t d/2)|^2 \left((|\bar{k}_l|^2 + k_\parallel^2) \frac{\sinh(\kappa_l d)}{2\kappa_l} - (|\bar{k}_l|^2 - k_\parallel^2) \frac{\sin(k_l d)}{2k_l} \right) \right. \\ & + |\bar{k}_t^2 - k_\parallel^2|^2 |\cos(\bar{k}_l d/2)|^2 \left((|\bar{k}_t|^2 + k_\parallel^2) \frac{\sinh(\kappa_t d)}{2\kappa_t} + (|\bar{k}_t|^2 - k_\parallel^2) \frac{\sin(k_t d)}{2k_t} \right) \\ & \left. - 4k_\parallel^2 |\cos(\bar{k}_l d/2)|^2 \left(\kappa_t (|\bar{k}_t|^2 + k_\parallel^2) \sinh(\kappa_t d) - k_t (|\bar{k}_t|^2 - k_\parallel^2) \sin(k_t d) \right) \right\} \end{aligned} \tag{22a}$$

$$\begin{aligned} N_a^{-2} &= A \left\{ 4|\bar{k}_t|^2 k_\parallel^2 |\sin(\bar{k}_t d/2)|^2 \left((|\bar{k}_l|^2 + k_\parallel^2) \frac{\sinh(\kappa_l d)}{2\kappa_l} + (|\bar{k}_l|^2 - k_\parallel^2) \frac{\sin(k_l d)}{2k_l} \right) \right. \\ & + |\bar{k}_t^2 - k_\parallel^2|^2 |\sin(\bar{k}_l d/2)|^2 \left((|\bar{k}_t|^2 + k_\parallel^2) \frac{\sinh(\kappa_t d)}{2\kappa_t} - (|\bar{k}_t|^2 - k_\parallel^2) \frac{\sin(k_t d)}{2k_t} \right) \\ & \left. - 4k_\parallel^2 |\sin(\bar{k}_l d/2)|^2 \left(\kappa_t (|\bar{k}_t|^2 + k_\parallel^2) \sinh(\kappa_t d) + k_t (|\bar{k}_t|^2 - k_\parallel^2) \sin(k_t d) \right) \right\} \\ & . \end{aligned} \tag{22b}$$

Equations (22) are very general. For the phonon modes (i.e. for real $k_\parallel$), $\bar{k}_t$ and $\bar{k}_l$ are either real or imaginary and the normalization constants (20) and (21) can be extracted from (22) by *taking the limit* of the redundant component of $\bar{k}_t$ and $\bar{k}_l$ going to zero. The normalization constants (19), (20), and (21) are chosen so that $||\mathbf{u}_{\mathbf{k}_\parallel, k_t, \sigma}|| = 1$, for any $\sigma, \mathbf{k}_\parallel$ and k_t. In the next section the wave-functions will be multiplied by still another constant, which will give the right dimensions to the phonon field.

4 Quantization of the elastic field

For the quantization of the elastic field we start from the classical Hamiltonian:

$$U = \int_V \left(\frac{\rho \dot{u}^2}{2} + \frac{S_{ij} c_{ijkl} S_{kl}}{2} \right) . \tag{23}$$

where S_{ij} are the components of the strain field, which is the *symmetric gradient* of the displacement field, $S_{ij} \equiv (\nabla_S \mathbf{u})_{ij} = (\partial_i u_j + \partial_j u_i)/2$. The canonical variables are the field, $\mathbf{u}$, and the conjugate momentum, $\boldsymbol{\pi} = \rho \mathbf{u}$, which satisfy the Hamilton equations

$$\dot{\mathbf{u}} = \frac{\delta U}{\delta \boldsymbol{\pi}} , \tag{24a}$$

$$\dot{\boldsymbol{\pi}} = -\frac{\delta U}{\delta \mathbf{u}} . \tag{24b}$$

Equation (24b) is nothing but the dynamic equation (1).

In the second quantization, $\mathbf{u}$ and $\boldsymbol{\pi}$ become the field operators, $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\pi}}$, respectively. If we denote by $\tilde{b}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}$ and $\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma}$, the creation and annihilation operators of a phonon with quantum numbers $\mathbf{k}_\parallel$, k_t and polarization σ (in the notation that we used before), then the *real* displacement and generalized momentum field operators, $\tilde{\mathbf{u}}(\mathbf{r}) = \tilde{\mathbf{u}}^\dagger(\mathbf{r})$ and $\tilde{\boldsymbol{\pi}}(\mathbf{r}) = \tilde{\boldsymbol{\pi}}^\dagger(\mathbf{r})$, are

$$\tilde{\mathbf{u}}(\mathbf{r}) = \sum_{\mathbf{k}_\parallel,k_t,\sigma} \left[\mathbf{f}_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma} + \mathbf{f}^*_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{b}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}\right] \tag{25a}$$

and

$$\begin{aligned}\tilde{\boldsymbol{\pi}}(\mathbf{r}) &= \rho \sum_{\mathbf{k}_\parallel,k_t,\sigma} \left[\dot{\mathbf{f}}_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma} + \dot{\mathbf{f}}^*_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{b}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}\right] \\ &= -i\rho \sum_{\mathbf{k}_\parallel,k_t,\sigma} \omega_{\mathbf{k}_\parallel,k_t,\sigma} \left[\mathbf{f}_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma} - \mathbf{f}_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma}\right]\end{aligned} \tag{25b}$$

where $\mathbf{f}_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r}) \equiv C\mathbf{u}_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})$ and C is a real constant which we shall determine from the commutation relations of the $\tilde{b}$ operators. In Eqs. (25) we do not take k_l as a summation variable, since this is either zero (for h fields), or is determined by $\mathbf{k}_\parallel$ and k_t (via Eq. (5a) or Eq. (5b) for the symmetric and antisymmetric case, respectively).

From Eqs. (25) we can extract the operators $\tilde{b}$ and $\tilde{b}^\dagger$ in terms of $\tilde{\mathbf{u}}$ and $\tilde{\boldsymbol{\pi}}$. In order to do this, let's first note from (4) that

$$[\mathbf{u}_{\mathbf{k}_\parallel,k_t,h}(\mathbf{r})]^* = \mathbf{u}_{-\mathbf{k}_\parallel,k_t,h}(\mathbf{r}), \tag{26a}$$

whereas

$$[\mathbf{u}_{\mathbf{k}_\parallel,k_t,s}(\mathbf{r})]^* = \mathbf{u}_{-\mathbf{k}_\parallel,k_t,s}(\mathbf{r}) \quad \text{or} \quad [\mathbf{u}_{\mathbf{k}_\parallel,k_t,s}(\mathbf{r})]^* = -\mathbf{u}_{-\mathbf{k}_\parallel,k_t,s}(\mathbf{r}), \tag{26b}$$

depending weather k_t is real or imaginary, respectively, and

$$[\mathbf{u}_{\mathbf{k}_\parallel,k_t,a}(\mathbf{r})]^* = \mathbf{u}_{-\mathbf{k}_\parallel,k_t,a}(\mathbf{r}) \quad \text{or} \quad [\mathbf{u}_{\mathbf{k}_\parallel,k_t,a}(\mathbf{r})]^* = -\mathbf{u}_{-\mathbf{k}_\parallel,k_t,a}(\mathbf{r}), \tag{26c}$$

depending weather k_l is real or imaginary, respectively. So

$$\int_V \mathbf{f}^T_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r}) \cdot \mathbf{f}_{\mathbf{k}'_\parallel,k'_t,\sigma'}(\mathbf{r})\, d^3\mathbf{r} = a_{\sigma,k_t,k_l} C^2 \delta_{\sigma,\sigma'}\delta_{\mathbf{k}_\parallel,-\mathbf{k}'_\parallel}\delta_{k_t,k'_t} \tag{27}$$

where by $\mathbf{f}^T$ we denote the transpose of the vector $\mathbf{f}$ and $a_{\sigma,k_t,k_l} = \pm 1$, according to Eqs. (26b) and (26c). Multiplying (25a) and (25b) by $\mathbf{f}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})$ and integrating over V, we get

$$\int_V \mathbf{f}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{\mathbf{u}}(\mathbf{r})\, d^3\mathbf{r} = C^2[\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma} + a_{\sigma,k_t,k_l}\tilde{b}^\dagger_{-\mathbf{k}_\parallel,k_t,\sigma}], \tag{28a}$$

$$\int_V \mathbf{f}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{\boldsymbol{\pi}}(\mathbf{r})\, d^3\mathbf{r} = -i\rho\omega_{\mathbf{k}_\parallel,k_t,\sigma}C^2\left[\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma} - a_{\sigma,k_t,k_l}\tilde{b}^\dagger_{-\mathbf{k}_\parallel,k_t,\sigma}\right] \tag{28b}$$

Solving the system we obtain

$$\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma} = \frac{1}{2C^2}\left[\int_V \mathbf{f}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{\mathbf{u}}(\mathbf{r})\, d^3\mathbf{r} + \frac{i}{\rho\omega_{\mathbf{k}_\parallel,k_t,\sigma}}\int_V \mathbf{f}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{\boldsymbol{\pi}}(\mathbf{r})\, d^3\mathbf{r}\right] \tag{29}$$

and obviously,

$$\tilde{b}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma} = \frac{1}{2C^2}\left[\int_V \mathbf{f}^T_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{\mathbf{u}}(\mathbf{r})\, d^3\mathbf{r} - \frac{i}{\rho\omega_{\mathbf{k}_\parallel,k_t,\sigma}}\int_V \mathbf{f}^T_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{\boldsymbol{\pi}}(\mathbf{r})\, d^3\mathbf{r}\right]. \tag{30}$$

Using the canonical commutation relations, $[\tilde{\mathbf{u}}(\mathbf{r}), \tilde{\mathbf{u}}(\mathbf{r}')] = [\tilde{\boldsymbol{\pi}}(\mathbf{r}), \tilde{\boldsymbol{\pi}}(\mathbf{r}')] = 0$ and $[\tilde{\mathbf{u}}(\mathbf{r}), \tilde{\boldsymbol{\pi}}(\mathbf{r}')] = i\hbar\delta(\mathbf{r}-\mathbf{r}')$, we obtain the commutation relations for the operators $\tilde{b}$ and $\tilde{b}^\dagger$:

$$[\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma}, \tilde{b}_{\mathbf{k}'_\parallel,k'_t,\sigma'}] = [\tilde{b}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}, \tilde{b}^\dagger_{\mathbf{k}'_\parallel,k'_t,\sigma'}] = 0$$

and

$$[\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma}, \tilde{b}^\dagger_{\mathbf{k}'_\parallel,k'_t,\sigma'}] = \delta_{\sigma,\sigma'}\delta_{\mathbf{k}_\parallel,\mathbf{k}'_\parallel}\delta_{k_t,k'_t},$$

provided that

$$C = \sqrt{\frac{\hbar}{2\rho\omega_{\mathbf{k}_\parallel,k_t,\sigma}}}\,. \tag{31}$$

Using Eqs. (25), with the proper normalization of $\mathbf{f}$, we can write U (23) in operator form,

$$\begin{aligned}
U \;=\; & \frac{\rho}{2}\int_V d^3\mathbf{r} \sum_{\mathbf{k}_\parallel,k_t,\sigma}\sum_{\mathbf{k}'_\parallel,k'_t,\sigma'}\left[\dot{\mathbf{f}}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{b}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma} + \dot{\mathbf{f}}^T_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma}\right] \\
& \times\left[\dot{\mathbf{f}}_{\mathbf{k}'_\parallel,k'_t,k'_l,\sigma'}(\mathbf{r})\tilde{b}_{\mathbf{k}'_\parallel,k'_t,k'_l,\sigma'} + \dot{\mathbf{f}}^\star_{\mathbf{k}'_\parallel,k'_t,k'_l,\sigma'}(\mathbf{r})\tilde{b}^\dagger_{\mathbf{k}'_\parallel,k'_t,k'_l,\sigma'}\right] \\
& +\frac{1}{2}\int_V d^3\mathbf{r} \sum_{\mathbf{k}_\parallel,k_t,\sigma}\sum_{\mathbf{k}'_\parallel,k'_t,\sigma'}\left[\partial_i[\mathbf{f}^\star_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})]_j\tilde{b}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma} + \partial_i[\mathbf{f}_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})]_j\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma}\right] \\
& \times c_{ijkl}\cdot\left[\partial_k[\mathbf{f}_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})]_l\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma} + \partial_k[\mathbf{f}^\star_{\mathbf{k}_\parallel,k_t,\sigma}(\mathbf{r})]_l\tilde{b}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}\right] \\
=\; & \sum_{\mathbf{k}_\parallel,k_t,\sigma}\hbar\omega_{\mathbf{k}_\parallel,k_t,\sigma}[\tilde{b}^\dagger_{\mathbf{k}_\parallel,k_t,\sigma}\tilde{b}_{\mathbf{k}_\parallel,k_t,\sigma} + 1/2]\,.
\end{aligned} \tag{32}$$

As expected, the Hamiltonian of the elastic body can be written as a sum of Hamiltonians of harmonic oscillators. These oscillators are the phonon modes of the plate.

We use this formalism elsewhere to describe the interaction of phonons with the disorder in amorphous materials [13, 14].

5 Conclusions

The vibrational modes of a thin plate (4) are well known from elasticity theory [11]. The purpose of the paper is to quantize the elastic field and for this we have to know if these modes, or part of them, form a complete set of orthogonal functions. But since the modes are the solutions of the eigenvalue-eigenvector problem of the operator $\tilde{L}$ (2), we showed that they form a complete set by proving that $\tilde{L}$ is self-adjoint.

Nevertheless, not all the functions of the form (4) are orthogonal to each-other, so to build the complete, orthogonal set of functions, we made use of a generic "momentum" operator, $\tilde{\mathbf{k}}_{\|} \equiv i(\partial_x + \partial_y)$, which commutes with $\tilde{L}$. Since for a plate with infinite lateral extension or a finite rectangular plate with periodic boundary conditions at the edges the operator $\tilde{\mathbf{k}}_{\|}$ is also self-adjoint, $\tilde{L}$ and $\tilde{\mathbf{k}}_{\|}$ admit a common, complete set of orthogonal eigenfunctions. The degenerate eigenvalues of $\tilde{\mathbf{k}}_{\|}$ are, of course, the wave-vectors parallel to the plate surfaces, $\mathbf{k}_{\|}$, of real components. Therefore, the complete set of eigenfunctions are the ones given by Eqs. (4), with real $\mathbf{k}_{\|}$.

In Section 2.2, based on the hermiticity of the operator $\tilde{L}$, we showed that these functions (4) are indeed orthogonal to each-other and in Section 3 we calculated the normalization factors.

Having all these ingredients, in Section 4 we presented the formal quantization procedure which is applied elsewhere [13, 14] to calculate the thermal properties of ultra-thin plates at low temperatures, and to deduce some of the observed features of the *standard tunneling model* in bulk amorphous materials.

6 Acknowledgements

We are grateful to Profs. N. Angelescu, Y. M. Galperin and M. Manninen for helpful discussions. DVA acknowledges the NATO support, by the grant EAP.RIG 982080.

References

[1] F. Giazotto, T. T. Heikkilä, A. Luukanen, A. M. Savin, and J. P. Pekola, Rev. Mod. Phys. **78** 217 (2006).

[2] M. M. Leivo, A. J. Manninen, and J. P. Pekola, Appl. Supercond. **5** 227 (1998).

[3] M. M. Leivo and J. P. Pekola, Appl. Phys. Lett. **72** 1305 (1998).

[4] D. V. Anghel, J. P. Pekola, M. M. Leivo, J. K. Suoknuuti, and M. Manninen, Phys. Rev. Lett. **81** 2958 (1998).

[5] D. V. Anghel and L. Kuzmin, Appl. Phys. Lett. **82** 293 (2003).

[6] L. S. Kuzmin, Physica B **284** 2129 (2000).

[7] A. Luukanen, *High Performance Microbolometers and Microcalorimeters: from 300 K to 100 mK*, PhD thesis, University of Jyväskylä, 2003.

[8] M. Leivo, *On-chip Cooling by Quasiparticle Tunneling Below 1 Kelvin*, PhD thesis, University of Jyväskylä, 1999.

[9] D. V. Anghel, *Phases and phase transitions in mesoscopic systems*, PhD thesis, University of Jyväskylä, 2000.

[10] H. Ezawa, Ann. Phys., NY **67** 438 (1971).

[11] B. A. Auld. *Acoustic Fields and Waves in Solids, 2nd Ed.*, Robert E. Krieger Publishing Company, 1990.

[12] F. Riesz and B. Sz. Nagy, *Functional Analysis (Translated from the 2^{nd} French edition by Leo F. Boron)*, 1956.

[13] D. V. Anghel, T. Kühn, Y. M. Galperin, and M. Manninen, submitted, cond-mat/0610469.

[14] T. Kühn, D. V. Anghel, and Y. M. Galperin, in preparation.